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Chiral Potts rapidity curve descended from six-vertex model and symmetry group of rapidities

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Abstract

In this paper, we present a systematical account of the descending procedure from the six-vertex model to the N-state chiral Potts model through fusion relations of $\tau^{(j)}$-operators, following the works of Bazhanov–Stroganov and Baxter–Bazhanov–Perk. A careful analysis of the descending process leads to the appearance of the high genus curve as the rapidity constraint for the chiral Potts models. Full symmetries of the rapidity curve are identified, as is its symmetry group structure. By normalized transfer matrices of the chiral Potts model, the $\tau^{(2)}T$ relation can be reduced to functional equations over a hyperelliptic curve associated with rapidities, by which the degeneracy of $\tau^{(2)}$-eigenvalues is revealed in the case of the superintegrable chiral Potts model.

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1. Introduction

The purpose of this paper is to revisit the known facts on the N-state chiral Potts model as a descendant of the six-vertex model and functional relations in the chiral Potts model. The discussion will mainly be based on two notable papers [14, 15] in this recently discovered solvable lattice model (for 'descendants' of a more general class of vertex models, see a recent work of Baxter [12]). The formulae appearing in this work are to a large extent borrowed from [14], and extensive use is also made of Baxter's other works. Hence, the present paper lays no claim to deep originality. In a way, our motivation is an attempt at better understanding the significance behind many identities in Baxter's papers, and clarifying the mathematical content of formulae appeared in [14]. However, after the analysis is made on the descending procedure, our effort leads to the appearance of the chiral Potts rapidity constraint in a natural way from the viewpoint of the descendant of the six-vertex model. Afterwards, we proceed to determine all symmetries of the rapidity curve, of which the large finite symmetry group structure has been widely believed for its role in solvability of the model; we further explore the degenerate eigenvalues of the six-vertex model through the chiral Potts transfer matrices, as an analogy to the discussion in [19] for the eight-vertex model for the root of unity cases. We therefore hope that the reader will still find our presentation to be of independent interest.

In the study of the two-dimensional solvable N-state chiral Potts model (for a brief history, see e.g., [22] section 4.1 and references therein), 'rapidities' of the statistical model are described by elements $[a, b, c, d]$ in the projective 3-space $\mathbf{P}^3$ satisfying the following equivalent sets of equations:

$$\mathfrak{W}: \quad \begin{cases} ka^N + k'c^N = d^N \\ kb^N + k'd^N = c^N \end{cases} \quad \Longleftrightarrow \quad \begin{cases} a^N + k'b^N = kd^N, \\ k'a^N + b^N = kc^N, \end{cases} \tag{1}$$

where k, k' are parameters with $k^2 + k'^2 = 1$, and $k' \neq \pm 1, 0$. The above relations define $\mathfrak{W}$ as an algebraic curve of genus $N^3 - 2N^2 + 1$, which will be called the rapidity curve throughout this paper. For simplicity, we shall confine our discussion of chiral Potts models only on the full homogeneous lattice by taking $p = p'$ in [14]. Note that a generalized column-inhomogeneous $\tau_2(t_q)$ model and its corresponding row-to-row transfer matrix functional relations without the conditions (1) have recently been discussed by Baxter in [13].

It is known in [15] that when descending the six-vertex model to the N-state chiral Potts model, one first solves the Yang–Baxter RLL relation of the six-vertex model to obtain the L-solution with operator entries acting on the 'quantum space' $\mathbf{C}^N$ (in the terminology of the quantum inverse scattering method, see e.g., [18, 20]), and with parameters depending on an arbitrary 4-vector ratio $p = [a, b, c, d] \in \mathbf{P}^3$ (for the explicit form, see formulae (6), (7) of this paper). The trace of the L-operator gives rise to a commuting family of operators $\tau_p^{(2)}(t)$ for $t \in \mathbf{C}$. Our attempt is to search certain principles which impose the rapidity constraint (1) for the chiral Potts model through descending processes from the six-vertex model. By examining the formulae of functional relations involved in the chiral Potts transfer matrices $T_p(q)$ in [14], we observe that the fusion relations of $\tau_p^{(j)}$-operators, which are induced from $\tau_p^{(2)}$, are equivalent to the rapidity constraint (1) on p. Then we go on to clarify all symmetries of the curve (1) and identify the structure of the automorphism group $\mathrm{Aut}(\mathfrak{W})$ through three hyperelliptic curves of genus $(N - 1)$ associated with $\mathfrak{W}$. One such hyperelliptic curve is defined by the variables $t = \frac{ab}{cd}$, $\lambda = \frac{d^N}{c^N}$ with the relation

$$W_{k'}: \quad t^N = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2}. \tag{2}$$

By normalizing the transfer matrices $T_p(q)$ as in [6], the $T\widehat{T}$ and $\tau^{(2)}T$ relations on $\mathfrak{W}$ can be reduced to functional equations of operators on $W_{k'}$. Then by using the (t, λ)-variable form of the $\tau^{(2)}T$ relation derived in this paper (which to the best of our knowledge has not previously appeared in the literature), we are able to show that the degeneracy of $\tau_p^{(2)}$-eigenvalues appears when p is the superintegrable element, in analogy to the discussion of the T–Q_{72} relation for the eight-vertex model in [19].

The remainder of this paper is organized as follows. In section 2, we begin our discussion by briefly reviewing the rapidities and Boltzmann weights of the N-state chiral Potts model in the literature (e.g., [1, 4]). In section 3, we start with a solution $\tau_p^{(2)}$ of the Yang–Baxter equation of the six-vertex model with the parameter $p \in \mathbf{P}^3$ in [15], then define $\tau_p^{(j)}$-operators for $0 \leqslant j \leqslant N$ through fusion relations, which originally appeared in the study of chiral Potts models in [14]. A careful analysis of the fusion relations of $\tau_p^{(j)}$, the constraint of p in rapidity curve $\mathfrak{W}$, naturally arises as a consequence of these relations. In section 4, we briefly review the results in [14] about the chiral Potts transfer matrices $T_p(q)$ for $p, q \in \mathfrak{W}$, $\tau^{(2)}T$ and $T\widehat{T}$ relations, and their connections with $\tau_p^{(j)}$. In this way, our $\tau^{(j)}$-fusion approach to the rapidity curve in the previous section can be better understood by the original source we are basing upon. In section 5, we first recall the relation of the rapidity curve of N-state chiral Potts model and three hyperelliptic curves of genus $(N-1)$ with D_N-symmetry in [7, 25]. Through this, we determine the full symmetries of the rapidity curve and its group structure. In particular, the symmetries that have already appeared in the literature (e.g., in [3, 4, 9]) exhaust all the symmetries of rapidities of the N-state chiral Potts model, and the order of the symmetry group $\mathrm{Aut}(\mathfrak{W})$ is equal to $4N^3$. In section 6, by using the normalized operator $V_p(t_q, \lambda_q)$ of $T_p(q)$ as in [6], we reduce the $\tau^{(2)}T$ and $T\widehat{T}$ relations from $\mathfrak{W}$ to functional equations on the hyperelliptic curve $W_{k'}$ in (2). Note that the (t,λ)-form of $T\widehat{T}$ relations was previously used in the effective discussions of 'ground-state' energy [6] and the excitation spectrum [23], both in the thermodynamic limit of an infinite lattice. Using the (t,λ)-form of the $\tau^{(2)}T$ relation, one can see the degenerate $\tau_p^{(2)}$-eigenvalues from the V_p-eigenvalues in the superintegrable case. We close in section 7 with some concluding remarks.

Notation. To present our work, we prepare some notation. In this paper, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ will denote the ring of integers, real, complex numbers, respectively, $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$ and $\mathrm{i} = \sqrt{-1}$. For $N \geqslant 2$, we fix the Nth root of unity,

$$\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}},$$

and $\mathbf{C}^N$ is the vector space consisting of all N-cyclic vectors with the basis $\{|n\rangle\}_{n\in\mathbf{Z}_N}$. For a positive integer n, we denote by $\otimes^n \mathbf{C}^N$ the tensor product of n copies of the vector space $\mathbf{C}^N$.

2. The rapidity curve of the N-state chiral Potts model

Let X, Z be the operators of $\mathbf{C}^N$ defined by $X|n\rangle = |n+1\rangle$, $Z|n\rangle = \omega^n|n\rangle$ for $n \in \mathbf{Z}_N$. Then X, Z satisfy the Weyl relation and Nth-power identity property: $XZ = \omega^{-1}ZX$, $X^N = Z^N = 1$.

We shall denote the 4-vector ratio by $[a, b, c, d]$ for a non-zero vector $(a, b, c, d) \in \mathbf{C}^4$. The collection of all 4-vector ratios is the projective 3-space $\mathbf{P}^3$. Hereafter, we shall always use the variables x, y, μ to denote the following component ratios of an element $[a, b, c, d] \in \mathbf{P}^3$,

$$x := \frac{a}{d}, \qquad y := \frac{b}{c}, \qquad \mu := \frac{d}{c}. \tag{3}$$

Then x, y, μ can be considered as affine coordinates of $\mathbf{P}^3$. From now on, we shall denote elements in $\mathbf{P}^3$ simply by $p, q, r, \ldots$, etc. The coordinates of an element, say p, will be written in the forms $a_p, b_p, x_p, \ldots$ so on, whenever it is necessary to specify the element p.

It is known that the rapidities of the N-state chiral Potts model form the projective curve $\mathfrak{W}$ (1) in $\mathbf{P}^3$. In terms of the affine coordinates (x, y, μ) in (3), an equivalent form of defining equations for $\mathfrak{W}$ is given by

$$kx^N = 1 - k'\mu^{-N}, \qquad ky^N = 1 - k'\mu^N, \qquad (x, y, \mu) \in \mathbf{C}^3. \tag{4}$$

Define

$$\mathrm{e}^{\mathrm{i}\theta_p} = \mathrm{e}^{\frac{-\pi\mathrm{i}}{N}} y_p, \qquad \mathrm{e}^{\mathrm{i}\phi_p} = x_p, \qquad u_p = \frac{N(\theta_p + \phi_p)}{2}, \qquad v_p = \frac{N(\theta_p - \phi_p)}{2}.$$

By eliminating the valuable μ^N in (4), $\mathfrak{W}$ becomes an N-fold unramified cover of the genus $(N-1)^2$ curve,

$$x^N + y^N = k(1 + x^N y^N) \quad (\text{equivalently}, \sin v_p = k \sin u_p). \tag{5}$$

The Boltzmann weights $W_{p,q}$, $\overline{W}_{p,q}$ of the N-state chiral Potts model are defined by the coordinates of $p, q \in \mathfrak{W}$ with the expressions

$$\frac{W_{p,q}(n)}{W_{p,q}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}$$
$$\times \left(= \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q} = \left(\frac{\cos N(\theta_q - \phi_p)/2}{\cos N(\theta_p - \phi_q)/2}\right)^{\frac{-n}{N}} \prod_{j=1}^{n} \frac{\sin\left(\frac{-\theta_q+\phi_p}{2} + \frac{\pi(2j-1)}{2N}\right)}{\sin\left(\frac{-\theta_p+\phi_q}{2} + \frac{\pi(2j-1)}{2N}\right)}\right),$$
$$\frac{\overline{W}_{p,q}(n)}{\overline{W}_{p,q}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}$$
$$\times \left(= (\mu_p \mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p} = \left(\frac{\sin N(\phi_q - \phi_p)/2}{\sin N(\theta_p - \theta_q)/2}\right)^{\frac{-n}{N}} \prod_{j=1}^{n} \frac{\sin\left(\frac{\phi_q-\phi_p}{2} + \frac{\pi(j-1)}{N}\right)}{\sin\left(\frac{\theta_p-\theta_q}{2} + \frac{\pi(j-1)}{N}\right)}\right)$$

(see, e.g., [2]). By the rapidity constraint (1), the above Boltzmann weights have the N-periodicity property for n. Equivalently, Boltzmann weights are represented by two cyclic vectors, $(W_{p,q}(n))_{n\in\mathbf{Z}_N}$ and $(\overline{W}_{p,q}(n))_{n\in\mathbf{Z}_N}$, of $\mathbf{C}^N$ with the ratio conditions:

$$\frac{W_{p,q}(n)}{W_{p,q}(n-1)} = \frac{d_p b_q - a_p c_q \omega^n}{b_p d_q - c_p a_q \omega^n}, \qquad \frac{\overline{W}_{p,q}(n)}{\overline{W}_{p,q}(n-1)} = \frac{\omega a_p d_q - d_p a_q \omega^n}{c_p b_q - b_p c_q \omega^n}.$$

For convenience, we shall assume $W_{p,q}(0) = \overline{W}_{p,q}(0) = 1$ without loss of generality.

3. Six-vertex model and fusion relations of τ^j

By a remarkable paper [15], Bazhanov and Stroganov found that with each element $[a, b, c, d] \in \mathbf{P}^3$, there is associated a solution $G(t)$ of the Yang–Baxter (YB) relation for the six-vertex model in terms of the operators X, Z. In the terminology of the quantum inverse scattering method, $G(t)$ is put into the following 2×2 matrix form with operator entries acting on 'quantum space' $\mathbf{C}^N$,

$$b^2 G(t) = b^2 \begin{pmatrix} G_{0,0} & G_{0,1} \\ G_{1,0} & G_{1,1} \end{pmatrix} = \begin{pmatrix} b^2 - td^2 X & (bc - \omega ad X)Z \\ -t(bc - ad X)Z^{-1} & -tc^2 + \omega a^2 X \end{pmatrix}, \qquad t \in \mathbf{C}, \qquad (6)$$

and satisfies the YB relation

$$R(t/t')(G(t) \bigotimes_{\text{aux}} 1)(1 \bigotimes_{\text{aux}} G(t')) = (1 \bigotimes_{\text{aux}} G(t'))(G(t) \bigotimes_{\text{aux}} 1)R(t/t'), \qquad (7)$$

where $R(t)$ is the following matrix[1] of 2-tensor of 'auxiliary space' $\mathbf{C}^2$,

$$R(t) = \begin{pmatrix} t\omega - 1 & 0 & 0 & 0 \\ 0 & t-1 & \omega - 1 & 0 \\ 0 & t(\omega - 1) & \omega(t-1) & 0 \\ 0 & 0 & 0 & t\omega - 1 \end{pmatrix}.$$

[1] The YB-relation solution $G(t)$ we describe here is in accordance with discussions in section 4 of [14], which will be briefly reviewed in section 4 of the present paper. Consequently, the R-matrix $R(x)$ in our content is required to vary the form appearing as (2.1) in [15]. Indeed, its entries are the Boltzmann weights of the six-vertex model described in (5) of [13], also previously discussed in [24].

By the auxiliary-space matrix product and quantum-space tensor product, the operator of a finite size L,

$$\bigotimes_{j=1}^{L} G_j(t) = G_1(t) \bigotimes \cdots \bigotimes G_L(t), \qquad G_j(t) := G(t), \tag{8}$$

again satisfies the YB relation (7), hence the traces,

$$\left\{ \mathrm{tr}_{\mathrm{aux}} \left(\bigotimes_{j=1}^{L} G_j(t) \right) \right\}_{t\in\mathbf{C}},$$

form a family of commuting operators of $\otimes^L \mathbf{C}^N$. As $G_j(t)$ depends on the parameter $p = [a, b, c, d] \in \mathbf{P}^3$, it will be written by $G_{p,j}(t)$ as well. Define the $\tau_p^{(2)}$-operator by

$$\tau_p^{(2)}(t) = \mathrm{tr}_{\mathrm{aux}} \left(\bigotimes_{j=1}^{L} G_{p,j}(\omega t) \right) \qquad \text{for} \quad t \in \mathbf{C}, \tag{9}$$

which again form a commuting family of operators acting on $\otimes^L \mathbf{C}^N$ for an arbitrary given $p \in \mathbf{P}^3$. The spin-shift operator of $\otimes^L \mathbf{C}^N$ will be denoted by X $\left(:= \bigotimes_{j=1}^{L} X_j\right)$, which has the eigenvalues ω^Q for $Q \in \mathbf{Z}_N$. In the study of chiral Potts transfer matrices in [14], there are families of operators, $\tau_p^{(j)}$ for $0 \leqslant j \leqslant N$, constructed from the $\tau_p^{(2)}$-family (9) by setting $\tau_p^{(0)}(t) = 0$, $\tau_p^{(1)}(t) = I$, and the following 'fusion relations' (see (4.27) of [14]):

$$\tau_p^{(j)}(t)\tau_p^{(2)}(\omega^{j-1}t) = z(\omega^{j-1}t)X\tau_p^{(j-1)}(t) + \tau_p^{(j+1)}(t), \qquad 1 \leqslant j \leqslant N, \tag{10}$$

$$\tau_p^{(N+1)}(t) := z(t)X\tau_p^{(N-1)}(\omega t) + u(t)I \tag{11}$$

with $z(t) := \left(\frac{\omega\mu_p^2(x_p y_p - t)^2}{y_p^4}\right)^L$, $u(t) := \alpha_p(\lambda) + \alpha_p(\lambda^{-1})$ where t, λ are related by (2), and

$$\alpha_p(\lambda) = \left(\frac{k'(1-\lambda_p\lambda)^2}{\lambda(1-k'\lambda_p)^2}\right)^L \left(= \left(\frac{\left(y_p^N - x^N\right)\left(t_p^N - t^N\right)}{y_p^{2N}\left(x_p^N - x^N\right)}\right)^L \text{ if } p \in \mathfrak{W} \right). \tag{12}$$

Note that $\alpha_p(\lambda) + \alpha_p(\lambda^{-1})$ can be expressed as a polynomial of $\lambda + \lambda^{-1}$, hence a polynomial of t^N.

By (10), one can express $\tau_p^{(j)}(t)$ for $j > 2$ as a 'polynomial' of $\tau_p^{(2)}$ of degree $(j-1)$ with coefficients in powers of X, e.g.,

$$\begin{aligned}
\tau_p^{(3)}(t) &= \tau_p^{(2)}(t)\tau_p^{(2)}(\omega t) - Xz(\omega t),\\
\tau_p^{(4)}(t) &= \tau_p^{(2)}(t)\tau_p^{(2)}(\omega t)\tau_p^{(2)}(\omega^2 t) - Xz(\omega t)\tau_p^{(2)}(\omega^2 t) - X\tau_p^{(2)}(t)z(\omega^2 t),\\
\tau_p^{(5)}(t) &= \tau_p^{(2)}(t)\tau_p^{(2)}(\omega t)\tau_p^{(2)}(\omega^2 t)\tau_p^{(2)}(\omega^3 t) - Xz(\omega t)\tau_p^{(2)}(\omega^2 t)\tau_p^{(2)}(\omega^3 t)\\
&\quad - X\tau_p^{(2)}(t)z(\omega^2 t)\tau_p^{(2)}(\omega^3 t) - Xz(\omega^3 t)\tau_p^{(2)}(t)\tau_p^{(2)}(\omega t) + X^2 z(\omega t)z(\omega^3 t).
\end{aligned}$$

Indeed, by induction argument one can show the following expressions of $\tau_p^{(j)}(t)$ for $2 \leqslant j \leqslant N+1$ in terms of $\tau_p^{(2)}$ and X:

$$\begin{aligned}
\tau_p^{(j)}(t) &= \prod_{s=0}^{j-2} \tau_p^{(2)}(\omega^j t) + \sum_{k=1}^{[\frac{j-1}{2}]} (-X)^k \sum_{1\leqslant i_1 <' i_2 <' \cdots <' i_k \leqslant j-2}\\
&\quad \times \prod_{\ell=1}^{k} \left(\frac{z(\omega^{i_\ell} t)}{\tau_p^{(2)}(\omega^{i_\ell - 1}t)\tau_p^{(2)}(\omega^{i_\ell} t)} \prod_{s=0}^{j-2} \tau_p^{(2)}(\omega^j t) \right)
\end{aligned} \tag{13}$$

where the notation $i_\ell <' i_{\ell+1}$ means $i_\ell + 1 < i_{\ell+1}$. Therefore, $\tau_p^{(j)}(t)$ commutes with $\tau_p^{(j')}(t')$ for all j, j', t, t'. However (11) imposes the constraint of $\tau_p^{(2)}(t)$, hence on p, of which the condition will be clear later on.

Using (13), one obtains the following relations:

$$\tau_p^{(N+1)}(t) = \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) + \sum_{k=1}^{[\frac{N}{2}]} (-X)^k \sum_{1 \leqslant i_1 <' i_2 <' \cdots <' i_k \leqslant N-1} \times \prod_{\ell=1}^{k} \left(\frac{z(\omega^{i_\ell} t)}{\tau_p^{(2)}(\omega^{i_\ell - 1} t) \tau_p^{(2)}(\omega^{i_\ell} t)} \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) \right);$$

$$-z(t) X \tau_p^{(N-1)}(\omega t) = \sum_{k=1}^{[\frac{N}{2}]} z(t)(-X)^k \sum_{0 = i_1 <' i_2 <' \cdots <' i_k \leqslant N-2} \times \prod_{\ell=1}^{k} \left(\frac{z(\omega^{i_\ell} t)}{\tau_p^{(2)}(\omega^{i_\ell - 1} t) \tau_p^{(2)}(\omega^{i_\ell} t)} \prod_{s=1}^{N-2} \tau_p^{(2)}(\omega^s t) \right).$$

By which the relations (10) and (11) give rise to the functional equation of $\tau_p^{(2)}(t)$:

$$F_{\tau_p^{(2)}}(t) = u(t) I, \tag{14}$$

where $F_{\tau_p^{(2)}}(t)$ is defined by

$$F_{\tau_p^{(2)}}(t) := \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) + \sum_{k=1}^{[\frac{N}{2}]} (-X)^k \sum_{I_k \in \mathcal{I}_k} \prod_{i \in I_k} \left(\frac{z(\omega^i t)}{\tau_p^{(2)}(\omega^{i-1} t) \tau_p^{(2)}(\omega^i t)} \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) \right), \tag{15}$$

with the index set $\mathcal{I}_k$ consisting of subsets I_k of $\mathbf{Z}_N$ with k distinct elements such that $i \not\equiv i' + 1 \pmod N$ for all $i, i' \in I_k$. For example, for $N = 2, 3, 4$ (15) is given by

$$N = 2, \qquad F_{\tau_p^{(2)}}(t) = \tau_p^{(2)}(t) \tau_p^{(2)}(-t) - (z(t) + z(-t)) X;$$

$$N = 3, \qquad F_{\tau_p^{(2)}}(t) = \prod_{j=0}^{2} \tau_p^{(2)}(\omega^j t) - \left(\sum_{j=0}^{2} z(\omega^j t) \tau_p^{(2)}(\omega^{j+1} t) \right) X;$$

$$N = 4, \qquad F_{\tau_p^{(2)}}(t) = \prod_{j=0}^{3} \tau_p^{(2)}(\omega^j t) - \left(\sum_{j=0}^{3} z(\omega^j t) \tau_p^{(2)}(\omega^{j+1} t) \right) X + (z(t) z(\omega^2 t) + z(\omega t) z(\omega^3 t)) X^2.$$

The functional equation (14) for $\tau_p^{(2)}(t)$, equivalently the relations (10) and (11), naturally imposes the constraint on p, and it turns out to be the requirement of p as an element in the rapidity curve $\mathfrak{W}$. Indeed, we have the following characterization of $\mathfrak{W}$.

Theorem 1. *For $p \in \mathbf{P}^3$, the relation (14) of $\tau_p^{(2)}(t)$ for $L = 1$ is equivalent to p being an element of the rapidity curve $\mathfrak{W}$.*

Proof. We now consider the relation (14) only for $L = 1$. By (6) and the expression of $u(t)$ in (11), we have

$$\begin{aligned} \tau_p^{(2)}(t) &= y_p^{-2} \left(\mu_p^2 \left(\omega x_p^2 - \omega t \right) X + \left(y_p^2 - \omega t \right) I \right), \\ u(t) &= \frac{(1 + k'^2)\left(1 + \mu_p^{2N}\right) - 4k' \mu_p^N}{\left(1 - k' \mu_p^N\right)^2} - \frac{(1 - k'^2)\left(1 + \mu_p^{2N}\right)}{\left(1 - k' \mu_p^N\right)^2} t^N. \end{aligned} \tag{16}$$

We are going to derive an explicit form of (15) for $L = 1$. By the invariant property under $t \mapsto \omega t$, $\prod_{j=0}^{N-1} \tau_p^{(2)}(\omega^j t)$ is expressed in powers of t^N; so is

$$\sum_{I_k \in \mathcal{I}_k} \prod_{i \in I_k} \left(\frac{z(\omega^i t)}{\tau_p^{(2)}(\omega^{i-1} t)\tau_p^{(2)}(\omega^i t)} \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) \right)$$

as the index set $\mathcal{I}_k$ is invariant under the translation by 1 (mod N). It is known that $X^N = I$, and $I, X, \ldots, X^{N-1}$ form a set of linearly independent matrices. By (16), one has

$$\prod_{j=0}^{N-1} \tau_p^{(2)}(\omega^j t) = c_0(p,t) + c_N(p,t) + \sum_{j=1}^{N-1} \binom{N}{j} c_j(p,t) X^j,$$

and

$$(-X)^k \sum_{I_k \in \mathcal{I}_k} \prod_{i \in I_k} \left(\frac{z(\omega^i t)}{\tau_p^{(2)}(\omega^{i-1} t)\tau_p^{(2)}(\omega^i t)} \prod_{s=0}^{N-1} \tau_p^{(2)}(\omega^s t) \right) = (-1)^k |\mathcal{I}_k| \sum_{j=k}^{N-k} \binom{N-2k}{j-k} c_j(p,t) X^j$$

for $1 \leqslant k \leqslant \left[\frac{N}{2}\right]$, where $c_j(p,t) := y_p^{-2N} \mu_p^{2j} \left(\omega^j x_p^{2j} y_p^{2N-2j} - t^N\right)$ for $0 \leqslant j \leqslant N$. Hence, $(15)_{L=1}$ has the following expression:

$$\begin{aligned} F_{\tau_p^{(2)}}(t) = {} & (c_0(p,t) + c_N(p,t)) + \sum_{j=1}^{[\frac{N}{2}]-1} \left(\sum_{k=0}^{j} (-1)^k |\mathcal{I}_k| \binom{N-2k}{j-k} \right) (c_j(p,t) X^j \\ & + c_{N-j}(p,t) X^{N-j}) + 2^{N-2[\frac{N}{2}]-1} \left(\sum_{k=0}^{[\frac{N}{2}]} (-1)^k |\mathcal{I}_k| \binom{N-2k}{[\frac{N}{2}]-k} \right) \\ & \times \left(c_{[\frac{N}{2}]}(p,t) X^{[\frac{N}{2}]} + c_{N-[\frac{N}{2}]}(p,t) X^{N-[\frac{N}{2}]} \right). \end{aligned} \tag{17}$$

Set $p = s := [\sqrt{t}, y\mu^{-1}, \mu^{-1}, 1]$ in (17) with generic t, y, μ. By (16) one has $\tau_s^{(2)}(t) = (1 - \omega y^{-2} t) I$, which implies $F_{\tau_s^{(2)}}(t)$ is a scalar operator, equivalently, the coefficients of X^j for $1 \leqslant j \leqslant N-1$ in (17) are all equal to zero. As $c_j(s,t) \neq 0$ for $j \geqslant 1$, one obtains the following recurrence relations for $|\mathcal{I}_k|$:

$$\sum_{k=0}^{j} (-1)^k |\mathcal{I}_k| \binom{N-2k}{j-k} = 0, \qquad j = 1, \ldots, \left[\frac{N}{2}\right], \tag{18}$$

by which (17) becomes the relation, $F_{\tau_p^{(2)}}(t) = c_0(p,t) + c_N(p,t)$ for $p \in \mathbf{P}^3$. By (16) and the expressions of $c_0(p,t)$ and $c_N(p,t)$, the functional relation $(14)_{L=1}$ can be reduced to the following equation involving only the scalar term:

$$y_p^{-2N}\left(y_p^{2N} + \mu_p^{2N} x_p^{2N} - t^N\left(1 + \mu_p^{2N}\right)\right) = \frac{(1+k'^2)\left(1+\mu_p^{2N}\right) - 4k'\mu_p^N}{\left(1 - k'\mu_p^N\right)^2} - \frac{(1-k'^2)\left(1+\mu_p^{2N}\right)}{\left(1-k'\mu_p^N\right)^2} t^N.$$

Then it is easy to see that the above relation is equivalent to the relations: $k^2 y_p^{2N} = \left(1 - k'\mu_p^N\right)^2$ and $k^2 x_p^{2N} = \left(1 - k'\mu_p^{-N}\right)^2$, i.e., p is an element of $\mathfrak{W}$ by (4). □

Remark. (1) As to the numerical values of $|\mathcal{I}_k|$, it is easy to see that $|\mathcal{I}_1| = N$ and $|\mathcal{I}_2| = \frac{N(N-3)}{2}$. However, for $k \geqslant 2$, it seems a non-trivial task to obtain the explicit form of $|\mathcal{I}_k|$ purely by the combinatoric method. Formula (18) provides a way to get the expression of $|\mathcal{I}_k|$ by recurrent relations. However, it would be interesting to have a certain combinatoric interpretation of (18).

(2) The fusion relations (10) and (11) were originally derived from the chiral Potts model with $p \in \mathfrak{W}$, which we will recall in the next section, hence the relation (14) holds for any site L when $p \in \mathfrak{W}$.

4. The $\tau^{(2)}T$ relation of the chiral Potts model

In this section, we recall the derivation of the fusion relations (10), (11) in the study of the chiral Potts models in [14]. For the N-state chiral Potts model on a lattice of horizontal size L with periodic boundary condition, the combined weights of intersection between two consecutive rows give rise to the transfer matrix acting on $\otimes^L \mathbf{C}^N$:

$$T_p(q)_{\sigma,\sigma'} = \prod_{l=1}^{L} \overline{W}_{p,q}(\sigma_l - \sigma'_l) W_{p,q}(\sigma_l - \sigma'_{l+1}), \qquad p, q \in \mathfrak{W}, \tag{19}$$

where $\sigma = (\sigma_1, \ldots, \sigma_L), \sigma' = (\sigma'_1, \ldots, \sigma'_L)$ with $\sigma_l, \sigma'_l \in \mathbf{Z}_N$. The Boltzmann weights satisfy the star–triangle relation:

$$\sum_{d=0}^{N-1} \overline{W}_{qr}(b-d) W_{pr}(a-d) \overline{W}_{pq}(d-c) = R_{pqr} W_{pq}(a-b) \overline{W}_{pr}(b-c) W_{qr}(a-c)$$

with $R_{pqr} = \frac{f_{pq} f_{qr}}{f_{pr}}$ and $f_{pq} = \left(\frac{\det_N(\overline{W}_{pq}(i-j))}{\prod_{n=0}^{N-1} W_{pq}(n)}\right)^{\frac{1}{N}}$ [4, 21], which ensures the commutativity of transfer matrices for a fixed $p \in \mathfrak{W}$:

$$[T_p(q), T_p(q')] = 0, \qquad q, q' \in \mathfrak{W}.$$

By (4), $T_p(q)$ depends only on the values of (x_q, y_q), parametrized by the curve (5) for p fixed. Hence, we shall also write $T_p(q)$ by $T_p(x_q, y_q)$ whenever it will be convenient. It is easy to see that $T_p(q)$ commutes with both the spin-shift operator X of $\otimes^L \mathbf{C}^N$ and the spatial translation operator S_R, which takes the jth column to the $(j+1)$th one for $1 \leqslant j \leqslant L$ with the identification $L+1 = 1$. We denote

$$\widehat{T}_p(q) = T_p(q) S_R.$$

Now we describe the $\tau^{(2)}T$ relation in [14, 15][2]. By following the arguments in section 4 of [14], one defines a 2×2 matrix, $G(g', g) = (G(g', g)_{m,m'})_{m,m'=0,1}$, for two vectors $g = \sum_k g(k)|k\rangle$, $g' = \sum_n g'(n)|n\rangle \in \mathbf{C}^N$ by

$$G(g', g) = \sum_{n,k} g'(n) g(k) G_n^k, \qquad G_n^k := G(|n\rangle, |k\rangle)$$

where G_n^k for $n, k \in \mathbf{Z}_N$ are given by (4.4), (3.37), (3.38) and (A.3) in [14]: $G_n^k = 0$ except $k = n, n-1$, and

$$G^n_{n\,m,m'} = (-1)^m \omega^{(m'-m)n+m} \left(\frac{c_p}{b_p}\right)^{m'+m} t_q^m,$$

$$G^{n-1}_{n\,m,m'} = (-1)^{m-1} \omega^{(m'-m)(n-1)+1} \left(\frac{d_p}{b_p}\right)^2 \left(\frac{a_p}{d_p}\right)^{m'+m} t_q^{1-m'},$$

where $t_q := x_q y_q$. Hence, $G^k_{n\,m,m'}$ can be put in the form of a 2×2 matrix with the operator valued acting on 'quantum space' $\mathbf{C}^N$. Indeed, one has the following expression:

$$\begin{pmatrix} G_{0,0} & G_{0,1} \\ G_{1,0} & G_{1,1} \end{pmatrix} = G(\omega t_q)$$

where $G(t)$ is defined in (6). For convenience, we shall hereafter denote the following component ratios of $p = [a, b, c, d] \in \mathbf{P}^3$ by

$$t(= t_p) := \frac{ab}{cd} = xy, \qquad \lambda(= \lambda_p) := \frac{d^N}{c^N} = \mu^N.$$

[2] $\tau_p^{(2)}(q)$, $T_p(q)$ in this paper are the operators $\tau^{(2)}_{k=0,q}$, T_q in [14], respectively.

By (4), the variables $(t, \lambda) = (t_p, \lambda_p)$ for $p \in \mathfrak{W}$ satisfy the relation (2), which defines a hyperelliptic curve of genus $N - 1$. By (9), we have the commuting family $\tau_p^{(2)}(t_q)$ for a given $p \in \mathfrak{W}$. The chiral Potts transfer matrices $T_p(q)$ constructed from the $\tau_p^{(2)}$-family in [14, 15] were along the line of the 'T–Q relation' developed in [5]. Apply the SL$_2$-gauge transform on the jth site $G_j(\omega t_q)$ in (8) in the following manner:

$$H_j = P_j^{-1} G_j(\omega t) P_{j+1}, \qquad P_j = \frac{1}{\sqrt{1+r_j^2}} \begin{pmatrix} 1 & r_j \\ -r_j & 1 \end{pmatrix}, \qquad (P_{L+1} := P_1),$$

then the trace remains the same, i.e., $\tau_p^{(2)}(t_q) = \mathrm{tr}_{\mathrm{aux}}(\bigotimes_j H_j)$. The choice of the P_j above is made in searching some non-trivial kernel vector $g_j \in \mathbf{C}^N$ of $H_{j;1,0}$ (the left lower entry of H_j) for each j so that one can construct Bethe-equation-type eigenvalues of $\tau_p^{(2)}(t_q)$. For $p, q \in \mathfrak{W}$, one can solve r_j, g_j as follows: for each basis element $\beta = \otimes_j |\beta_j\rangle \in \bigotimes^L \mathbf{C}^N$ with $\beta_j \in \mathbf{Z}_N$, there is associated a set of solutions r_j and the kernel vectors g_j, given by (4.14), (4.19a) in [14]:

$$r_j^\beta = \omega^{1-\beta_{j-1}} x_q, \qquad g_j^\beta(n) = {y_p}^2 \overline{W}_{p,Uq}(n - \beta_{j-1}) W_{p,Uq}(n - \beta_j), \qquad \beta_0 := \beta_L,$$

where U is the following automorphism of $\mathfrak{W}$:

$$U : \mathfrak{W} \longrightarrow \mathfrak{W}, [a, b, c, d] \mapsto [\omega a, b, c, d], \qquad ((x, y, \mu) \mapsto (\omega x, y, \mu)).$$

Furthermore, the vectors $g_j'^\beta = H_{j;0,0}(g_j^\beta)$ and $g_j''^\beta = H_{j;1,1}(g_j^\beta)$ have the expressions

$$g_j'^\beta(n) = \frac{(y_p - \omega x_q)(t_p - t_q)}{x_p - x_q} \overline{W}_{p,q}(n - \beta_{j-1}) W_{p,q}(n - \beta_j),$$

$$g_j''^\beta(n) = \frac{\omega\mu_p^2(x_p - \omega x_q)(t_p - \omega t_q)}{y_p - \omega^2 x_q} \overline{W}_{p,U^2q}(n - \beta_{j-1} - 1) W_{p,U^2q}(n - \beta_j - 1),$$

(see (4.19b), (4.19c) in [14]). This implies

$$\tau_p^{(2)}(t_q)\left(\otimes_j g_j^\beta\right) = \otimes_j g_j'^\beta + \otimes_j g_j''^\beta.$$

By the expression of $T_p(q)$ in (19), the above $g_j^\beta(n)$, $g_j'^\beta(n)$ and $g_j''^\beta(n)$ for all basis elements β give rise to the following $\tau^{(2)}T$ relation (4.20) in [14] (or (14) in [11]):

$$\tau_p^{(2)}(t_q) T_p(\omega x_q, y_q) = \varphi_p(q) T_p(x_q, y_q) + \overline{\varphi}_p(Uq) X T_p(\omega^2 x_q, y_q), \tag{20}$$

where

$$\varphi_p(q) := \left(\frac{(y_p - \omega x_q)(t_p - t_q)}{y_p^2(x_p - x_q)}\right)^L, \qquad \overline{\varphi}_p(q) := \left(\frac{\omega\mu_p^2(x_p - x_q)(t_p - t_q)}{y_p^2(y_p - \omega x_q)}\right)^L.$$

By which, one can express $\tau_p^{(2)}(t_q)$ in terms of T_p:

$$\tau_p^{(2)}(t_q) = (\varphi_p(q) T_p(x_q, y_q) + \overline{\varphi}_p(Uq) X T_p(\omega^2 x_q, y_q)) T_p(\omega x_q, y_q)^{-1}. \tag{21}$$

The commutativity of $T_p(q)$ ensures that $\tau_p^{(2)}(t_q)$ commutes with $T_p(x_{q'}, y_{q'})$,

$$\left[\tau_p^{(2)}(t_q), T_p(x_{q'}, y_{q'})\right] = 0, \qquad \text{for} \quad p, q, q' \in \mathfrak{W}.$$

The $\tau_p^{(j)}$ in (10), (11), with $t = t_q$, are related to the transfer matrices $T_p(q)$ by the following $T\widehat{T}$ relations ((3.46) for $(l, k) = (j, 0)$ in [14], or (13) in [11]):

$$T_p(x_q, y_q) \widehat{T}_p(y_q, \omega^j x_q) = r_{p,q} h_{j;p,q} \left(\tau_p^{(j)}(t_q) + \frac{z(t_q) z(\omega t_q) \cdots z(\omega^{j-1} t_q)}{\alpha_p(\lambda_q)} X^j \tau_p^{(N-j)}(\omega^j t_q)\right) \tag{22}$$

for $0 \leqslant j \leqslant N$, where

$$r_{p,q} = \left(\frac{N(x_p - x_q)(y_p - y_q)\left(t_p^N - t_q^N\right)}{\left(x_p^N - x_q^N\right)\left(y_p^N - y_q^N\right)(t_p - t_q)} \right)^L,$$

$$h_{j;p,q} = \left(\prod_{m=1}^{j-1} \frac{y_p^2(x_p - \omega^m x_q)}{(y_p - \omega^m x_q)(t_p - \omega^m t_q)} \right)^L$$

and $\alpha_p(\lambda_q)$ is defined in (12). In particular, the relation (22) for $j = N$ becomes

$$T_p(x_q, y_q)\widehat{T}_p(y_q, x_q) = \left(\frac{N y_p^{2N-2}(y_p - y_q)(y_p - x_q)}{\left(y_p^N - y_q^N\right)\left(y_p^N - x_q^N\right)} \right)^L \tau_p^{(N)}(t_q) \tag{23}$$

(see (4.39), (4.44) in [14]). Indeed, the operators $\tau_p^{(j)}$ were originally defined by the relation (22) in [14], and the fusion relations of $\tau^{(j)}$ were derived from (20) and (22) with the coefficient $z(t)$ in (11) satisfying $z(t_q) = \varphi_p(q)\overline{\varphi}_p(q)$. Using (21) and (10), one can successively express $\tau_p^{(j)}(q)$ for $1 \leqslant j \leqslant N+1$ in terms of $T_p(q)$ ((4.34) in [14]):

$$\begin{aligned}\tau_p^{(j)}(q) = T_p(x_q, y_q)T_p(\omega^j x_q, y_q) \sum_{m=0}^{j-1} (\varphi_p(q)\varphi_p(Uq)\cdots\varphi_p(U^{m-1}q)\overline{\varphi}_p(U^{m+1}q)\cdots \\ \times \overline{\varphi}_p(U^{j-1}q)T_p(\omega^m x_q, y_q)^{-1}T_p(\omega^{m+1}x_q, y_q)^{-1}X^{j-m-1}).\end{aligned} \tag{24}$$

Then, by $\prod_{j=0}^{N-1} \varphi_p(U^j q) = \alpha_p(\lambda_q)$ and $\prod_{j=0}^{N-1} \overline{\varphi}_p(U^j q) = \alpha_p\left(\lambda_q^{-1}\right)$, the relation (11) automatically follows. In this way, one can interpret the expression of $\tau_p^{(2)}$ in (21), or equivalently the $\tau^{(2)}T$ relation (20), as a $\tau_p^{(2)}$-solution of the functional equation (14) when p is an element of $\mathfrak{W}$. By (23) and the T_p-expression of $\tau_p^{(N)}(q)$ in (24), one obtains the functional equation of chiral Potts transfer matrices T_p ((4.40) of [14]):

$$\widehat{T}_p(y_q, x_q) = \sum_{m=0}^{N-1} C_{m;p}(q)T_p(x_q, y_q)T_p(\omega^m x_q, y_q)^{-1}T_p(\omega^{m+1}x_q, y_q)^{-1}X^{-m-1},$$

where

$$\begin{aligned}C_{m;p}(q) = \varphi_p(q)\varphi_p(Uq)\cdots\varphi_p(U^{m-1}q)\overline{\varphi}_p(U^{m+1}q)\cdots \\ \times \overline{\varphi}_p(U^{N-1}q) \left(\frac{N y_p^{2N-2}(y_p - y_q)(y_p - x_q)}{\left(y_p^N - y_q^N\right)\left(y_p^N - x_q^N\right)} \right)^L.\end{aligned}$$

5. The symmetry group of chiral Potts rapidity curve and its relation with hyperelliptic curves with D_N-symmetry

It is known that the rapidities of the chiral Potts model have a large finite symmetry group. In this section, we are going to identify the precise group structure of Aut($\mathfrak{W}$). As in [9], we consider the following automorphisms of $\mathfrak{W}$:

$$\begin{array}{ll} M^{(1)} : [a,b,c,d] \mapsto [\omega a, b, c, \omega d], & (x,y,\mu) \mapsto (x, y, \omega\mu), \\ M^{(2)} : [a,b,c,d] \mapsto [\omega a, \omega b, c, d], & (x,y,\mu) \mapsto (\omega x, \omega y, \mu), \\ M^{(3)} : [a,b,c,d] \mapsto \left[c, \omega^{\frac{1}{2}} d, \omega^{\frac{-1}{2}} a, \omega^{-1} b\right], & (x,y,\mu) \mapsto \left(\omega y^{-1}, \omega x^{-1}, \omega^{\frac{-1}{2}} x^{-1} y \mu^{-1}\right), \\ M^{(4)} : [a,b,c,d] \mapsto [a, b, \omega^{-1} c, d], & (x,y,\mu) \mapsto (x, \omega y, \omega\mu), \\ M^{(5)} : [a,b,c,d] \mapsto \left[d, \omega^{\frac{1}{2}} c, \omega^{\frac{-1}{2}} b, a\right], & (x,y,\mu) \mapsto \left(x^{-1}, \omega y^{-1}, \omega^{\frac{1}{2}} x y^{-1} \mu\right), \\ R : [a,b,c,d] \mapsto [b, \omega a, d, c], & (x,y,\mu) \mapsto (y, \omega x, \mu^{-1}). \end{array} \tag{25}$$

Then one has

$$R = M^{(3)}M^{(5)} = M^{(2)}M^{(5)}M^{(3)}, \qquad R^2 = M^{(2)}, \qquad {M^{(3)}}^2 = {M^{(5)}}^2 = {M^{(4)}}^N = 1.$$

The diagonal symmetries of coordinates of $\mathfrak{W}$ are expressed by

$$\begin{aligned}
&U = M^{(1)}M^{(2)}{M^{(4)}}^{-1} : [a, b, c, d] \mapsto [\omega a, b, c, d];\\
&{M^{(1)}}^{-1}M^{(4)} : [a, b, c, d] \mapsto [a, \omega b, c, d];\\
&{M^{(4)}}^{-1} : [a, b, c, d] \mapsto [a, b, \omega c, d];\\
&{M^{(2)}}^{-1}M^{(4)} : [a, b, c, d] \mapsto [a, b, c, \omega d].
\end{aligned}$$

Note that $\mathfrak{W}/\langle M^{(1)}\rangle$ is represented by the curve (5), of which for the $N = 3$ case, the Picard–Fuch equation of periods and algebraic geometry properties of the theta function and the Jacobian variety were investigated in detail in [16, 17, 21]. Denote

$$\begin{aligned}
M^{(0)} = M^{(1)}M^{(2)}{M^{(4)}}^{-2} &: [a, b, c, d] \mapsto [\omega a, b, \omega c, d], \quad \text{equivalently,}\\
&(x, y, \mu) \mapsto (\omega x, \omega^{-1}y, \omega^{-1}\mu).
\end{aligned}$$

Among the three morphisms $M^{(i)}$ for $i = 0, 1, 2$, any two automorphisms generate a $\mathbf{Z}_N^2$-group acting freely on $\mathfrak{W}$. Their quotient Riemann surfaces can be realized as members in the following one-parameter family of hyperelliptic curves:

$$W_\kappa\ (= W_{N,\kappa}) : T^N = \frac{(1-\kappa\Lambda)(1-\kappa\Lambda^{-1})}{1-\kappa^2}, \qquad (T, \Lambda) \in \mathbf{C}^2,$$

where κ is a complex parameter $\neq 0, \pm 1$. For $N \geqslant 3$, the above family of curves is characterized by the hyperelliptic curves of genus $N-1$ with $\mathbf{Z}_2 \times D_N$ symmetry group, where D_N is the dihedral group (see proposition 2 in [26]). The hyperelliptic involution is given by

$$\sigma : (T, \Lambda) \mapsto (T, \Lambda^{-1}),$$

and D_N is generated by the automorphisms θ, ι of order N, 2, respectively,

$$\theta : (T, \Lambda) \mapsto (\omega T, \Lambda), \qquad \iota : (T, \Lambda) \mapsto \left(\frac{1}{T}, \frac{1-\kappa\Lambda}{\kappa-\Lambda}\right).$$

It is known that the three N^2-unramified quotients of $\mathfrak{W}$ can be realized as the following hyperelliptic curves ((25) in [25]):

$$W_{k'} \simeq \mathfrak{W}/\langle M^{(0)}, M^{(1)}\rangle, \qquad W_{\mathrm{i}k'/k} \simeq \mathfrak{W}/\langle M^{(1)}, M^{(2)}\rangle, \qquad W_k \simeq \mathfrak{W}/\langle M^{(0)}, M^{(2)}\rangle, \tag{26}$$

with the coordinate expression from $\mathfrak{W}$ to hyperelliptic curves given by

$$\begin{aligned}
&\mathfrak{W} \longrightarrow W_{k'}, && [a, b, c, d] \mapsto (t, \lambda) = \left(\frac{ab}{cd}, \frac{d^N}{c^N}\right);\\
&\mathfrak{W} \longrightarrow W_{\mathrm{i}k'/k}, && [a, b, c, d] \mapsto (T_r, \Lambda_r) = \left(\frac{ac}{bd}, \frac{\mathrm{i}d^N}{b^N}\right);\\
&\mathfrak{W} \longrightarrow W_k, && [a, b, c, d] \mapsto (T_l, \Lambda_l) = \left(\omega^{\frac{1}{2}}\frac{bc}{ad}, \frac{d^N}{a^N}\right),
\end{aligned} \tag{27}$$

((4), (9), (11) in [25])[3]. The hyperelliptic curve W_κ can also be represented in the following different form [25, 26]. By quotients of symmetries of W_κ, one has the following commutative diagram of Riemann surfaces:

$$\begin{array}{ccccc} & W_\kappa & \xrightarrow{\Psi} & \mathbf{P}^1 & = W_\kappa/\langle\theta\rangle \\ & \downarrow \Pi & & \downarrow \pi & \\ W_\kappa/\langle\sigma\rangle = & \mathbf{P}^1 & \xrightarrow{\psi} & \mathbf{P}^1 & = W_\kappa/\langle\theta,\sigma\rangle \end{array}$$

where Ψ, ψ, Π, π are the natural projections with the coordinate expressions:

$$\Psi(T,\Lambda) = \lambda, \qquad \Pi(T,\Lambda) = T, \qquad \psi(T) = T^N, \qquad \pi(\Lambda) = \frac{(1-\kappa\Lambda)(1-\kappa\Lambda^{-1})}{1-\kappa^2}.$$

The (T,Λ)-coordinates in W_κ of branch points for the projections Ψ and Π are given by

$$\text{Branch points of } \Psi: (\infty, 0), \quad (\infty,\infty), \quad (0,\kappa), \quad (0,\kappa^{-1}),$$

$$\text{Branch points of } \Pi: \left(\omega^{-j}\sqrt[N]{\frac{1+\kappa}{1-\kappa}}, -1\right), \quad \left(\omega^{-j}\sqrt[N]{\frac{1-\kappa}{1+\kappa}}, 1\right), \quad 1 \leqslant j \leqslant N,$$

where $\sqrt[N]{\frac{1-\kappa}{1+\kappa}} := \sqrt[N]{\left|\frac{1-\kappa}{1+\kappa}\right|}\,\mathrm{e}^{\frac{\mathrm{i}}{N}\arg(\frac{1-\kappa}{1+\kappa})}$. Using the birational transformations,

$$w = \frac{\kappa}{1-\kappa^2}\left(\Lambda - \frac{1}{\Lambda}\right), \qquad \Lambda = \frac{1}{2\kappa}\{(1-\kappa^2)(w - T^N) + \kappa^2 + 1\},$$

one obtains the equivalent form of the curve W_κ in terms of (w, T)-variables,

$$W_\kappa : w^2 = \left(T^N - \frac{1-\kappa}{1+\kappa}\right)\left(T^N - \frac{1+\kappa}{1-\kappa}\right), \qquad (T, w) \in \mathbf{C}^2.$$

Now we are able to determine all the symmetries of the rapidity curve $\mathfrak{W}$ and its group structure through the hyperelliptic curve $W_{\mathrm{i}k'/k}$. By (26) and (27), $\mathfrak{W}$ is an unramified cover over $W_{\mathrm{i}k'/k}$ with the $\mathbf{Z}_N^2$-covering group $\langle M^{(1)}, M^{(2)}\rangle$ via the map

$$\xi : \mathfrak{W} \longrightarrow W_{\mathrm{i}k'/k}, \qquad [a,b,c,d] \mapsto (T,\Lambda) = \left(\frac{ac}{bd}, \frac{\mathrm{i}d^N}{b^N}\right),$$

by which the symmetries of $W_{\mathrm{i}k'/k}$ can be lifted to automorphisms of $\mathfrak{W}$ in the following manner:

$$M^{(3)} \to \sigma; \qquad M^{(4)} \to \theta^{-1}; \qquad R \to \iota\cdot\theta; \qquad M^{(5)} \to \iota\cdot\theta\cdot\sigma. \tag{28}$$

Proposition 1. *For $N \geqslant 3$, the automorphism group* $\mathrm{Aut}(\mathfrak{W})$ *of $\mathfrak{W}$ is generated by $M^{(j)}$, $1 \leqslant j \leqslant 5$, and we have the following exact sequence of groups:*

$$1 \longrightarrow \mathbf{Z}_N^2 \longrightarrow \mathrm{Aut}(\mathfrak{W}) \longrightarrow \mathbf{Z}_2 \times D_N \longrightarrow 1.$$

As a consequence, the order of $\mathrm{Aut}(\mathfrak{W})$ *is equal to* $4N^3$.

Proof. When $\frac{k'}{k} \neq \pm 1$, $\mathrm{Aut}(W_{\mathrm{i}k'/k})$ is generated by σ, θ, ι with its structure isomorphic to $\mathbf{Z}_2 \times D_N$ [26]. As σ, θ, ι can be lifted to those of $\mathfrak{W}$ via (28), one has the surjective group homomorphism from $\mathrm{Aut}(\mathfrak{W})$ onto $\mathrm{Aut}(W_{\mathrm{i}k'/k})$ with the kernel isomorphic to $\mathbf{Z}_N^2$. Then the result follows. When $\frac{k'}{k} = \pm 1$, by replacing $W_{\mathrm{i}k'/k}, M^{(2)}$ by $W_{k'}, M^{(0)}$, respectively, the same argument again gives the conclusion for $\mathrm{Aut}(\mathfrak{W})$. □

[3] The variables $(T_r, \Lambda_r), (T_l, \Lambda_l)$ here and $(t_r, \lambda), (t_l, \lambda)$ in equations (9), (11) of [25] are related by $(T_r, \Lambda_r) = (t_r^{-1}, \frac{\mathrm{i}k\lambda}{1-k'\lambda}), (T_l, \Lambda_l) = (\omega^{\frac{1}{2}} t_l, \frac{-k\lambda}{k'-\lambda})$.

6. The descended forms of $\tau^{(2)}T$ and $T\widehat{T}$ relations on the hyperelliptic curve

In the study of chiral Potts models, for the $T_p(q)$-eigenvalue problem one reduces the operators on $\mathfrak{W}$ to those over $W_{k'}$ [6, 23]; while discussions of the order parameter problem of chiral Potts models were conducted by using the curve $W_{\mathrm{i}k'/k}$ [8–10]. In this section, we consider only the formal case, and derive the functional equations on $W_{k'}$ corresponding to the $\tau^{(2)}T$ and $T\widehat{T}$ relations on $\mathfrak{W}$. By $T_p(q) = T_p(x_q, y_q)$ and $W_{k'} = \mathfrak{W}/\langle M^{(0)}, M^{(1)}\rangle$ in (26), for the reduction of $T_p(q)$ to an operator on $W_{k'}$, one needs only to examine the effect of $T_p(q)$ when replacing q by $M^{(0)}(q)$. The relation is given by formula (2.40) in [14]:

$$T_p(\omega x_q, \omega^{-1} y_q) = \left(\frac{(y_p - \omega x_q)(y_p - \omega^{-1} y_q)}{\mu_p^2(\omega x_p - y_q)(x_p - x_q)}\right)^L X^{-1} T_p(x_q, y_q). \tag{29}$$

In order to eliminate the scalar factor in the above right-hand side, a procedure of normalizing $T_p(x_q, y_q)$ was given in [6] via the function $g_p(q)\overline{g}_p(q)$, where $g_p, \overline{g}_p$ are functions on $\mathfrak{W}$ defined by

$$g_p(q) := \prod_{n=0}^{N-1} W_{pq}(n) \qquad \left(= \left(\frac{\mu_p}{\mu_q}\right)^{\frac{(N-1)N}{2}} \prod_{j=1}^{N-1} \left(\frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}\right)^{N-j} \right),$$

$$\overline{g}_p(q) := \det{}_N(\overline{W}_{pq}(i-j))$$
$$\left(= N^{\frac{N}{2}} \mathrm{e}^{\frac{\pi\mathrm{i}(N-1)(N-2)}{12}} \prod_{j=1}^{N-1} \frac{(t_p - \omega^j t_q)^j}{(x_p - \omega^j x_q)^j (y_p - \omega^j y_q)^j}, \text{ by (2.44) in [14]} \right).$$

One has

$$g_p(q)\overline{g}_p(q) = N^{\frac{N}{2}} \mathrm{e}^{\frac{\pi\mathrm{i}(N-1)(N-2)}{12}} \left(\frac{\mu_p}{\mu_q}\right)^{\frac{(N-1)N}{2}} \prod_{k=1}^{N-1} \frac{(x_p - \omega^k y_q)^k (t_p - \omega^k t_q)^k}{(x_q - \omega^k y_p)^k (x_p - \omega^k x_q)^k (y_p - \omega^k y_q)^k}. \tag{30}$$

By $\mu_p^N(x_p^N - x_q^N)(x_p^N - y_q^N) = \mu_p^{-N}(y_p^N - x_q^N)(y_p^N - y_q^N)$, one obtains the following relation for the function $g\overline{g}$ when changing the variable q to $M^{(0)}(q)$:

$$g_p(M^{(0)}q)\overline{g}_p(M^{(0)}q) = (-1)^{N-1} \left(\frac{(y_p - \omega x_q)(y_p - \omega^{-1} y_q)}{\mu_p^2(\omega x_p - y_q)(x_p - x_q)}\right)^N g_p(q)\overline{g}_p(q). \tag{31}$$

By comparing the factors in (29) and (31), one leads to the operator

$$V_p(q) = S_R^{\frac{-1}{2}} T_p(q)/(g_p(q)\overline{g}_p(q))^{\frac{L}{N}}, \tag{32}$$

with the size L being only *even for N even*[4]. By $X^N = 1$ and the relation between $\mathfrak{W}$ and $W_{k'}$, the operator $V_p(q)^N$ depends only on the values of t_q and λ_q, hence is defined on the curve $W_{k'}$. Up to Nth roots of unity, we may write $V_p(q)$ simply by $V_p(t_q, \lambda_q)$. With the same argument, one can see that $\prod_{j=0}^{N-1} V_p(\omega^j t_q, \lambda_q)$ depends on the values of t_q^N, λ_q, hence becomes a function of the variable λ_q only. Indeed, by examining poles of the function, Baxter obtained its precise form ((4) of [6]):

$$\prod_{j=0}^{N-1} V_p(\omega^j t_q, \lambda_q) = \eta^L \lambda_q^{\frac{-(N-1)L}{2}} \alpha_p(\lambda_q)^{\frac{-(N-1)}{2}} S(\lambda_q), \qquad \eta := \mathrm{e}^{\frac{\pi\mathrm{i}(N-1)(N+4)}{12}}, \tag{33}$$

[4] This requirement was not put in [6], but we add it here out of consideration of the factor $(-1)^{N-1}$ in the relation (31).

where $\alpha_p(\lambda)$ is given by (12), and $S(\lambda)$ is a polynomial of λ of degree $(N-1)L$. We shall use the operator $V_p(t_q, \lambda_q)$ on $W_{k'}$ to describe the $\tau^{(2)}T$ and $T\widehat{T}$ relations.

The $T\widehat{T}$ relations on $W_{k'}$ were already given in [6] as equation (8) there. For the self-contained nature of this paper, we represent here a little more detailed derivation on the formula by using (22). By (29), one has

$$\widehat{T}_p(\omega^j y_q, x_q) = \left(\frac{1}{\mu_p^{2j}}\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)(y_p-\omega^{j-k}x_q)}{(\omega x_p-\omega^{j+1-k}x_q)(x_p-\omega^{k-1}y_q)}\right)^L X^{-j}\widehat{T}_p(y_q, \omega^j x_q),$$

by which (22) can be converted into the following form:

$$\begin{aligned} T_p(x_q, y_q)\widehat{T}_p(\omega^j y_q, x_q) &= r_{p,q}h_{j;p,q}\left(\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)(y_p-\omega^{j-k}x_q)}{\mu_p^2(\omega x_p-\omega^{j+1-k}x_q)(x_p-\omega^{k-1}y_q)}\right)^L \\ &\quad\times\left(X^{-j}\tau_p^{(j)}(t_q)+\frac{\prod_{k=1}^{j}z(\omega^{k-1}t_q)}{\alpha_p(\lambda_q)}\tau_p^{(N-j)}(\omega^j t_q)\right). \end{aligned}$$

By (30), one can derive the identity

$$\begin{aligned} g_p(q)\overline{g}_p(q)g_p(U^{j+1}R^{-1}q)\overline{g}_p(U^{j+1}R^{-1}q) &= \frac{N^N\lambda_p^{N-1}\left(x_p^N-y_q^N\right)^j(y_p-y_q)^N(y_p-x_q)^N}{\mathrm{e}^{\frac{\pi\mathrm{i}(N-1)(N-2)}{6}}\left(y_p^N-y_q^N\right)^{N+j}\left(y_p^N-x_q^N\right)^N} \\ &\quad\times\left(\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)^N}{(x_p-\omega^{k-1}y_q)^N(t_p-\omega^{k-1}t_q)^N}\right)\left(t_p^N-t_q^N\right)^j\prod_{k=1}^{N-1}(t_p-\omega^{k+j}t_q)^{2k}. \end{aligned}$$

Then by the relation

$$\begin{aligned} &(r_{p,q}h_{j;p,q})^{\frac{1}{L}}\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)(y_p-\omega^{j-k}x_q)}{\mu_p^2(\omega x_p-\omega^{j+1-k}x_q)(x_p-\omega^{k-1}y_q)} \\ &\quad=\frac{Ny_p^{2j-2}(y_p-x_q)(y_p-y_q)\left(t_p^N-t_q^N\right)}{\omega^j\mu_p^{2j}\left(x_p^N-x_q^N\right)\left(y_p^N-y_q^N\right)}\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)}{(x_p-\omega^{k-1}y_q)(t_p-\omega^{k-1}t_q)}, \end{aligned}$$

one has

$$\begin{aligned} &\frac{(r_{p,q}h_{j;p,q})^{\frac{N}{L}}\left(\prod_{k=1}^{j}\frac{(y_p-\omega^k y_q)(y_p-\omega^{j-k}x_q)}{\mu_p^2(\omega x_p-\omega^{j+1-k}x_q)(x_p-\omega^{k-1}y_q)}\right)^N}{g_p(q)\overline{g}_p(q)g_p(U^{j+1}R^{-1}q)\overline{g}_p(U^{j+1}R^{-1}q)} \\ &\quad=\frac{\eta^2}{\prod_{k=1}^{N-1}\left[\omega\mu_p^2y_p^{-4}(t_p-\omega^{j+k}t_q)^2\right]^k}\left(\frac{\left(y_p^N-x_q^N\right)\left(t_p^N-t_q^N\right)}{y_p^{2N}\left(x_p^N-x_q^N\right)}\right)^{N-j}. \end{aligned}$$

By which, the $T\widehat{T}$ relations (22) become equation (8) of [6] in variables (t,λ),

$$\alpha_p(\lambda)^{\frac{j}{N}}\zeta(\omega^j t)V_p(t,\lambda)V_p(\omega^j t,\lambda^{-1}) = \alpha_p(\lambda)X^{-j}\tau_p^{(j)}(t)+\left(\prod_{k=1}^{j}z(\omega^{k-1}t)\right)\tau_p^{(N-j)}(\omega^j t),$$

where $\zeta(t) := \eta^{\frac{-2L}{N}}\prod_{k=1}^{N-1}z(\omega^k t)^{\frac{k}{N}}$. In particular for $j=0$, we have $\tau_p^{(N)}(t)=\zeta(t)V_p(t,\lambda)V_p(t,\lambda^{-1})$. Then by (33), one arrives at formulae (11), (12) in [6],

$$\begin{aligned} &S(\lambda)S(\lambda^{-1}) = \tau_p^{(N)}(t)\tau_p^{(N)}(\omega t)\cdots\tau_p^{(N)}(\omega^{N-1}t), \\ &V_p(t,\lambda)^N = \frac{\eta^L}{\lambda^{\frac{(N-1)L}{2}}\alpha_p(\lambda)^N S(\lambda^{-1})}\prod_{j=1}^{N}\left(\alpha_p(\lambda)X^{-j}\tau_p^{(j)}(t)+\left(\prod_{k=1}^{j}z(\omega^{k-1}t)\right)\tau_p^{(N-j)}(\omega^j t)\right). \end{aligned} \quad (34)$$

By the commutativity of operators X, $\tau_p^{(j)}(t)$, $V_p(t,\lambda)$ and $S(\lambda)$, their eigenvalues again satisfy the relations (10), (11), (34), regarded as scalar functions on $W_{k'}$. By which, one can solve first $\tau_p^{(j)}(t)$ by (10), (11), then by (34) obtain $S(\lambda)$, hence eigenvalues of $V_p(t,\lambda)$ (equivalently, those of $T_p(q)$). All the above relations should in principle place one well on road to solving the eigenvalue problem of chiral Potts model; however as (t,λ) are the 'coordinates' of a higher genus curve $W_{k'}$, it is still a difficult problem to extract explicit solutions for a finite site L. Nevertheless, one can use these equations to obtain the maximum eigenvalues [6] in the thermodynamic limit as L tends to ∞, as well as in the discussion of the excitation spectrum in [23].

We now identify the (t,λ)-form of $\tau^{(2)}T$ relation. By (30), the relation (20) becomes

$$
\begin{aligned}
y_p^{2L}&\left(\prod_{k=1}^{N-1}\frac{(t_p-\omega^{k+1}t_q)^k}{(\omega x_q-\omega^k y_p)^k(x_p-\omega^{k+1}x_q)^k}\right)^{\frac{L}{N}}\tau_p^{(2)}(t_q)V_p(\omega t_q,\lambda_q)\\
&=(-\omega)^L\left(\frac{(x_q-\omega^{-1}y_p)^N(t_p-t_q)^N}{(x_p-x_q)^N}\prod_{k=1}^{N-1}\frac{(t_p-\omega^k t_q)^k}{(x_q-\omega^k y_p)^k(x_p-\omega^k x_q)^k}\right)^{\frac{L}{N}}\\
&\quad\times V_p(t_q,\lambda_q)+(-\omega)^{-L}\left(\frac{\lambda_p^2(x_p-\omega x_q)^N(t_p-\omega t_q)^N}{(x_q-\omega^{-2}y_p)^N}\right.\\
&\quad\left.\times\prod_{k=1}^{N-1}\frac{(t_p-\omega^{k+2}t_q)^k}{(\omega^2x_q-\omega^k y_p)^k(x_p-\omega^{k+2}x_q)^k}\right)^{\frac{L}{N}}XV_p(\omega^2t_q,\lambda_q),
\end{aligned}
$$

hence one has

$$
\begin{aligned}
y_p^{2L}\tau_p^{(2)}(t_q)V_p(\omega t_q,\lambda_q)&=\left(\frac{(-1)^N\omega^{N(N+1)/2}\left(x_q^N-y_p^N\right)\left(t_p^N-t_q^N\right)}{\left(x_p^N-x_q^N\right)}\right)^{\frac{L}{N}}V_p(t_q,\lambda_q)\\
&\quad+\lambda_p^{\frac{2L}{N}}(t_p-\omega t_q)^{2L}\left(\frac{\left(x_p^N-x_q^N\right)}{(-1)^N\omega^{N(N+1)/2}\left(x_q^N-y_p^N\right)\left(t_p^N-t_q^N\right)}\right)^{\frac{L}{N}}XV_p(\omega^2t_q,\lambda_q).
\end{aligned}
$$

By $(-\omega)^N\omega^{N(N-1)/2}=-1$ and $\frac{(y_p^N-x_q^N)}{(x_p^N-x_q^N)}=\frac{\lambda_p\lambda_q-1}{\lambda_p^{-1}\lambda_q-1}$, we obtain the $\tau^{(2)}V$ relation on the variable $(t_q,\lambda_q)\in W_{k'}$ for a fixed $(t_p,\lambda_p)\in W_{k'}$:

$$
\begin{aligned}
\left(\frac{1-k'\lambda_p}{k}\right)^{\frac{2L}{N}}\tau_p^{(2)}(t_q)V_p(\omega t_q,\lambda_q)&=\left(\frac{\left(\lambda_p^2\lambda_q-\lambda_p\right)\left(t_p^N-t_q^N\right)}{\lambda_q-\lambda_p}\right)^{\frac{L}{N}}V_p(t_q,\lambda_q)\\
&\quad+(t_p-\omega t_q)^{2L}\left(\frac{\lambda_p\lambda_q-\lambda_p^2}{(\lambda_p\lambda_q-1)\left(t_p^N-t_q^N\right)}\right)^{\frac{L}{N}}XV_p(\omega^2t_q,\lambda).
\end{aligned}
\tag{35}
$$

In particular, in the superintegrable case where $\lambda_p=1, t_p=\left(\frac{1-k'}{1+k'}\right)^{\frac{1}{N}}$, the relation (35) becomes

$$
\begin{aligned}
\left(\frac{1-k'}{1+k'}\right)^{\frac{L}{N}}\tau_p^{(2)}(t_q)V_p(\omega t_q,\lambda_q)&=\left(\frac{1-k'}{1+k'}-t_q^N\right)^{\frac{L}{N}}V_p(t_q,\lambda_q)\\
&\quad+\left(\left(\frac{1-k'}{1+k'}\right)^{\frac{1}{N}}-\omega t_q\right)^{2L}\left(\frac{1-k'}{1+k'}-t_q^N\right)^{\frac{-L}{N}}XV_p(\omega^2t_q,\lambda_q).
\end{aligned}
$$

Therefore, if $\tau_p^{(2)}(t_q)$ coupling with a function $V_p(t_q,\lambda_q)$ of $W_{k'}$ forms a solution of the above relation, $V_p\left(t_q,\lambda_q^{-1}\right)$ is also a solution with the same $\tau_p^{(2)}(t_q)$. By the procedure of solving the eigenvalue $V_p(t_q, \lambda_q)$ in the chiral Potts model, $V_p(t_q, \lambda_q)$ is not invariant under the change of λ_q to λ_q^{-1}. Hence, the correspondence between the V_p-eigenvalues and the $\tau_p^{(2)}$-eigenvalues is at least a 2–1 map. Therefore, the degeneracy of $\tau_p^{(2)}$-eigenvalues occurs when p is the superintegrable element, an analogy to the discussion of the T–Q_{72} relation for the eight-vertex model in [19].

7. Concluding remarks

In this paper, we made a clear mathematical derivation of the descending process from the six-vertex model to the chiral Potts N-state model following the works [14, 15]. We start with the Yang–Baxter solution (6) of the six-vertex model carrying an arbitrary 4-vector ratio p; then reinterpret the descendant relation of the six-vertex model and chiral Potts model through the fusion relations (10), (11) of $\tau_p^{(j)}$, and finally reach the chiral Potts constraint (1) for the rapidity p. The finding does suggest that studies of all $\tau^{(j)}$-families should be important for understanding the mathematics in the chiral Potts transfer matrices. Although the operators $\tau^{(j)}$ in statistical mechanics are in many respects well understood physically, the mathematical investigation on the fusion relations of these operators still lags behind. Certain interesting topics are expected to arise by exploring deeper into their mathematical structures. From the relations between the rapidity curve $\mathfrak{W}$ and three genus $(N-1)$ hyperelliptic curves with D_N-symmetry, we determined the structure of Aut($\mathfrak{W}$), hence all the symmetries of rapidities. Through one of these hyperelliptic curves, $W_{k'}$ in (2), we obtain the reduced form (35) of the $\tau^{(2)}T$ relation on $W_{k'}$, which is descended from $\mathfrak{W}$. Through this, we are able to indicate the degeneracy of $\tau_p^{(2)}$-eigenvalues when p is a superintegrable point, a similar phenomenon for the T–Q_{72} relation discussion of the eight-vertex model in [19]. The comparison is also one of the motivations for our investigation in this paper. Further developments along this line are now under consideration, and progress is expected.

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